ON CONJUGACY SEPARABILITY OF FUNDAMENTAL GROUPS
OF GRAPHS OF GROUPS

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ABSTRACT. A complete determination of when the elements of a fundamental group of a (countable) graph of profinite groups are conjugacy distinguished is given. By embedding an arbitrary fundamental group G into one with profinite vertex groups and making use of the above result, questions on conjugacy separability of G can be reduced to the solution of equations in the vertex groups of G.

1. INTRODUCTION

An element g of a group G is said to be *conjugacy distinguished* (or conjugacy separable) in G if for every element h of G not conjugate to g there exists $N \triangleleft_f G$ such that gN and hN are nonconjugate in G/N. A group is *conjugacy separable* if all its elements are conjugacy distinguished. The best-known classes of conjugacy separable groups are polycyclic-by-finite groups [10], profinite groups [11], free-by-finite groups [5], and certain Fuchsian groups [19]. Results are also known on conjugacy separability of certain amalgamated free products (e.g., of free groups [18]), and some one-relator groups with torsion [1]. In the last mentioned paper the authors ask whether every one-relator group with torsion is conjugacy separable. More generally, one can ask when the fundamental group of a graph of groups is conjugacy separable.

In order to investigate these problems one might adopt the following strategy: let G be a residually finite group, with $\widehat{G}$ its profinite completion (i.e., $\widehat{G}$ is the inverse limit of the system of finite images G/N of G). It is easily shown that if $g, h \in G$ are conjugate in every finite image of G, then they are conjugate in $\widehat{G}$, and so G is conjugacy separable if and only if whenever $g, h \in G$ are conjugate in $\widehat{G}$, then they are conjugate in G. Unfortunately, if G is a nontrivial fundamental group of a graph of groups, then $\widehat{G}$ is still not sufficiently well understood (cf. [7, 20]) to allow for the successful completion of the last step. A complete answer can be given when the vertex groups are themselves profinite groups. This, in turn, allows necessary and sufficient conditions to be derived for the general case.

Let X be a connected graph, let $\mathscr{G} = \{A_x, \ x \in VX, \ H_e, \ e \in EX\}$ be a graph of groups over X, and let $G = \pi_1(\mathscr{G}, X)$ be the fundamental group of

this graph of groups (see [2, 4], or [14] for the relevant definitions and properties). We assume that G is residually finite, and denote by $\widehat{A}_x$ the profinite completion of the vertex group A_x with respect to the induced topology $\{A_x \cap M : M \triangleleft_f G\}$. Let $\overline{H}_e$ denote the topological closure of H_e in $\widehat{A}_{i(e)}$. The edge isomorphisms can be extended to these closures (cf. Lemma 3.1). The result is a graph of groups over X with fundamental group G^+ (for details see §3). The obvious maps $A_x \to \widehat{A}_x$ extend to an embedding of G into G^+. If we denote the conjugacy class of g in G by g^G and $\bigcap_{M \triangleleft_f G} g^G M$ by $\mathrm{cl}_G(g^G)$, then we have:

Theorem 3.4. *For every $g \in G$ we have $\mathrm{cl}_G(g^G) = \mathrm{cl}_{G^+}(g^{G^+}) \cap G$.*

The point of this result is that most elements of G^+ turn out to be conjugacy distinguished (i.e, $\mathrm{cl}_{G^+}(g^{G^+}) = g^{G^+}$). If $g \in G$ is such an element, then g is conjugacy distinguished in G if and only if $g^{G^+} \cap G = g^G$. This condition is quite tractable, and amounts to whether certain equations hold in the vertex groups A_x and $\widehat{A}_x$. Assuming that we have enough information about the vertex groups and their completions, the conjugacy separability of G can be decided.

To state the results for G^+, let Y be a maximal subtree of X, and write $A^+ = \langle \widehat{A}_x : x \in VX \rangle \leq G^+$. Then A^+ is the fundamental group of the graph of groups $\mathscr{G}$ restricted to Y, and G^+ is an HNN-extension with base group A^+. As a consequence of 4.4, 4.6, and 4.9 below we get

Theorem A. *Let $a \in A^+$.*
   (i) *If either no conjugate of a belongs to a vertex group, or X is locally finite, $a \in \widehat{A}_x$, and no $\widehat{A}_x$-conjugate of a belongs to an edge subgroup, then a is conjugacy distinguished in A^+.*
   (ii) *Assume that X is locally finite, let $D = \bigcup_{e \in EX} \overline{H}_e$, and suppose $a \in D$ belongs to only a finite number of edge subgroups. Then a is conjugacy distinguished in A^+ if and only if there exist vertices $x_1, \ldots, x_m$ of Y such that $D \cap a^{A^+} = D \cap \{a^{a_1 \cdots a_m} : a_i \in \widehat{A}_{x_i}\}$.* □

For elements of G^+ we obtain from 5.1, 5.3, 5.6, and 5.8

Theorem B. (i) *All elements of $G^+ \backslash A^+$, and all elements of A^+ satisfying condition* (i) *of Theorem* A, *are conjugacy distinguished in G^+.*
   (ii) *Assume that X is locally finite, and let $D' = \bigcup_{e \in EX \backslash EY} \overline{H}_e$ and $C = \bigcup_{x \in VX} \widehat{A}_x$.*
   Then $h \in D'$ is conjugacy distinguished in G^+ if and only if $h^{G^+} \cap C$ is a union of a finite number of sets of the form $C \cap \{h^g : g \in \widehat{A}_{x_1} t_{e_1}^{\varepsilon_1} \cdots \widehat{A}_{x_n} t_{e_n}^{\varepsilon_n} \widehat{A}_{x_{n+1}}$, for some $e_i \in EX \backslash EY$, $\varepsilon_i = \pm 1$, and $x_i \in VX\}$.

This in particular settles the problem for the case when the vertex groups are profinite, since such a G can be residually finite if and only if $G \cong G^+$, by 3.3 below.

2. Notation

Let X be a directed connected graph, $\mathscr{G}$ a graph of groups $\{A_x : x \in VX ; H_e : e \in EX$ with isomorphisms θ_e from $H_e \leq A_{i(e)}$ to $H_{\bar{e}} \leq A_{l(e)}\}$ over

X, and $G = \pi_1(\mathscr{G}, X)$ the fundamental group of $(\mathscr{G}, X)$ relative to the choice of a fixed maximal subtree Y (see [2, 4], or [14] for details). It is well known that if $A = \pi_1(\mathscr{G}|_Y, Y)$ then the obvious map of A to $\langle A_x : x \in VY \rangle \subseteq G$ is an isomorphism, and G is an HNN-extension with base group A and stable letters t_e corresponding to the edges $e \in EX \backslash EY$:

$$G = \langle A, t_e : t_e^{-1} h t_e = h\theta_e \text{ for all } h \in H_e, \ e \in EX \backslash EY \rangle .$$

In the notation of [16] let $I = I(\mathscr{G}, X)$ denote the set of all sequences $(P_x)_{x \in VX}$ satisfying the following conditions: (a) $P_x \lhd A_x$ and there exists an integer $m = m(P)$ such that $|A_x : P_x| \leq m$ for all $x \in VX$, (b) $(P_{i(e)} \cap H_e)\theta_e = P_{t(e)} \cap H_{\bar{e}}$ for all $e \in EX$. For $P, Q \in I$ we write $P \leq Q$ if $P_x \subseteq Q_x$ for all x. For $P \in I$ let $\mathscr{G}_P$ denote the graph of groups $\{A_x/P_x, H_e P_{i(e)}/P_{i(e)}$, induced isomorphisms $\theta_{e,P}\}$, and let $G_P = \pi_1(\mathscr{G}_P, X)$. The projections $A_x \to A_x/P_x$ clearly extend to an epimorphism $\pi_P : G \to G_P$. It is well known that G_P is free-by-finite (e.g., [14, Exercise 2 on p. 123]). Also note that if $M \lhd_f G$ then $P = (M \cap A_x)_{x \in VX} \in I$, and G/M is a homomorphic image of G_P. In particular, this implies that if G is residually finite then $\bigcap_{P \in I} P_x = \langle 1 \rangle$ for all $x \in VX$. We assume this from now on. For $g \in G$ write g^G for the conjugacy class of g in G.

Proposition 2.1. *Let* $g \in G$. *Then* $\bigcap_{M \lhd_f G} g^G M = \bigcap_{P \in I} g^G \ker \pi_P$. *In particular,* $\bigcap_{M \lhd_f G} M = \bigcap_{P \in I} \ker \pi_P$.

Proof. If $M \lhd_f G$ induces $P \in I$ then $M \supseteq \ker \pi_P$, so $g^G M \supseteq \bigcap_P g^G \ker \pi_P$. Conversely, given $P \in I$ let $P \uparrow = \{M \lhd_f G : M \cap A_x = P_x \text{ for all } x \in VX\}$. Now by a theorem of Dyer [5], G_P is conjugacy separable, which implies that

$$(g\pi_P)^{G_P} = \bigcap_{M \in P\uparrow} (g\pi_P)^{G_P} M\pi_P = \bigcap_{M \in P\uparrow} (g^G M)\pi_P ,$$

whence $g^G \ker \pi_P = \bigcap_{M \in P\uparrow} g^G M$. The result follows. (For the final part take $g = 1$.) $\square$

3. PROFINITE CLOSURES

It is evident that if $P, Q \in I$, then $P \cap Q = (P_x \cap Q_x)_{x \in VX}$ also belongs to I. We refer to the topology on A_x with $\{P_x : P \in I\}$ as a fundamental system of open neighbourhoods of the identity as the I-topology. For each x let $\widehat{A}_x = \varprojlim_{P \in I} (A_x/P_x)$, the inverse limit being formed relative to the partial order $\leq$ of I introduced above. Let $\overline{H}_e$ denote the topological closure of H_e in $\widehat{A}_{i(e)}$. We have

Lemma 3.1. *For every* $e \in EX$, *the isomorphism* $\theta_e : H_e \to H_{\bar{e}}$ *extends to an isomorphism* $\overline{\theta}_e : \overline{H}_\varepsilon \to \overline{H}_{\bar{e}}$ *such that* $(\overline{H}_e \cap \overline{P}_{i(e)})\overline{\theta}_e = \overline{H}_{\bar{e}} \cap \overline{P}_{t(e)}$.

Proof. For each $P \in I$ there are canonical isomorphisms

$$\overline{H}_e/(\overline{H}_e \cap \overline{P}_{i(e)}) \cong \overline{H}_e \overline{P}_{i(e)}/\overline{P}_{i(e)} = H_e \overline{P}_{i(e)}/\overline{P}_{i(e)}$$
$$\cong H_e/(H_e \cap \overline{P}_{i(e)}) = H_e/H_e \cap P_{i(e)} ,$$

and $\overline{H}_e$ is naturally isomorphic to the inverse limit $\varprojlim_{P \in I} (H_e/H_e \cap P_{i(e)})$ [12]. The isomorphisms induced by θ_e on the quotients $H_e/H_e \cap P_{i(e)}$ are compatible

with the inverse limit structure, so the existence of $\overline{\theta}_e$ follows. (If one thinks of the inverse limit $\widehat{A}_x$ as a subgroup of the cartesian product $\prod_{x \in VX}(A_x/P_x)$, then the image of the element $h = (h_P\pi_P) \in \overline{H}_e$ is $h\overline{\theta}_e = (h_P\theta_e\pi_P) \in \overline{H}_{\overline{e}}$.) The intersection property of $\overline{\theta}_e$ is also trivial from the construction. □

Thus we have a graph of groups $\mathscr{G}^+ = \{\widehat{A}_x, \overline{H}_e, \overline{\theta}_e\}$ over X. Let $G^+ = \pi_1(\mathscr{G}^+, X)$. The natural embeddings $A_x \to \widehat{A}_x$ extend to a homomorphism $\mu: G \to G^+$. The next few results exploit the relationship between G^+ and the residual properties of G.

Lemma 3.2. *Let G, G^+, and μ be as above. Then G^+ is residually finite and $\ker\mu = \bigcap_{M \triangleleft_f G} M$.*

Proof. The edge subgroups of G^+ are closed in their vertex groups by definition, and 3.1 implies that if $P \in I(\mathscr{G})$ then $\overline{P} = (\overline{P}_x) \in I(\mathscr{G}^+)$. The residual finiteness of G^+ is therefore a consequence of the theorem of [16]. For the second part let $P \in I$, and consider the following diagram with exact top row:

$$\begin{array}{ccc}
\ker\mu \rightarrowtail G & \xrightarrow{\ \mu\ } & G^+ \\[4pt]
{\scriptstyle\pi_P}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi_P^+} \\[4pt]
G_P & \xrightarrow{\ \mu_P\ } & G_{\overline{P}}^+
\end{array}$$

The diagram is easily seen to commute, and $A_x/P_x \cong \widehat{A}_x/\overline{P}_x$ implies that μ_P is an isomorphism. Then $1 = (\ker\mu)\mu\pi_P^+ = (\ker\mu)\pi_P\mu_P$ implies that $\ker\mu \subseteq \ker\pi_P$ for all P, so $\ker\mu \subseteq \bigcap_P \ker\pi_P = \bigcap_{M \triangleleft_f G} M$ by 2.1. On the other hand, $G/\ker\mu$ is a subgroup of G^+; being residually finite, the reverse inclusion follows. □

Before proceeding further we mention that, in general, profinite closedness of the edge subgroups in their vertex groups is not necessary for the residual finiteness of G (cf. [15, 17]). In the case of profinite vertex groups, however, we have the following result.

Corollary 3.3. *Let $G = \pi_1(\mathscr{G}, X)$, and assume that $A_x = \varprojlim_{P \in I}(A_x/P_x)$ for all $x \in VX$. Then G is residually finite if and only if every edge subgroup is profinitely closed in its vertex group.*

Proof. Suppose there exists an element $a \in \overline{H}_e \backslash H_e$ for some $e \in EX$. If e is an edge in the maximal subtree Y let $g = a^{-1}(a\overline{\theta}_e)$; otherwise let $g = a^{-1}t_e(a\overline{\theta}_e)t_e$. Then $g \neq 1$ in G, while $g\mu = 1$, so G is not residually finite by 3.2. The converse follows from 3.2 and the fact that when the edge subgroups are closed we have $G = G^+$. □

If S is a subset of a group H, write $\mathrm{cl}_H(S)$ for $\bigcap_{M \triangleleft_f H} SM$.

Theorem 3.4. *Let G, G^+, and μ be as above. If $g \in G$ then $\mathrm{cl}_G(g^G)\mu = \mathrm{cl}_{G^+}((g\mu)^{G^+}) \cap G\mu$.*

We are not assuming that μ is injective.

Proof. Let $S = \operatorname{cl}_G(g^G) = \bigcap_{P \in I} g^G \ker \pi_P$ (by 2.1). Then using $\ker \mu \subseteq \ker \pi_P$ (cf. the proof of 3.2) we have

$$S\mu = \bigcap_{P \in I} (g^G \ker \pi_P)\mu = \bigcap_{P \in I} (g\mu)^{G\mu}(\ker \pi_P)\mu.$$

Now it is easy to see from the commutative diagram in the proof of 3.2 and the fact that μ_P is an isomorphism, that $(\ker \pi_P)\mu = \ker \pi_P^+ \cap G\mu$. The proof of 2.1 applies without change to show that $\operatorname{cl}_{G^+}(y^{G^+}) = \bigcap_{P \in I} y^{G^+} \ker \pi_P^+$ for all $y \in G^+$. Thus

$$S\mu = \bigcap_{P \in I} (g\mu)^{G\mu}(\ker \pi_P^+ \cap G\mu) = \bigcap_{P \in I} (g\mu)^{G\mu} \ker \pi_P^+ \cap G\mu,$$

which is certainly contained in $\bigcap_{P \in I}(g\mu)^{G^+} \ker \pi_P^+ \cap G\mu$. Conversely, suppose we have an element $z\mu \in \bigcap_{P \in I}(g\mu)^{G^+} \ker \pi_P^+$, where $z \in G$. Then for each P there exists $w_P \in G^+$ such that $z\mu\pi_P^+ = (w_P^{-1}(g\mu)w_P)\pi_P^+$. Since μ_P is an isomorphism, there exists $u_P \in G$ such that $u_P\pi_P\mu_P = w_P\pi_P^+$. The above equation now becomes $z\pi_P\mu_P = (u_P^{-1}gu_P)\pi_P\mu_P$, and since μ_P is injective we obtain $z \in \bigcap_{P \in I} g^G \ker \pi_P = S$. We have now shown that

$$S\mu = \bigcap_{P \in I} (g\mu)^{G\mu} \ker \pi_P^+ \cap G\mu \subseteq \bigcap_{P \in I}(g\mu)^{G^+} \ker \pi_P^+ \cap G\mu \subseteq S\mu,$$

as required. □

Since conjugacy separable groups are residually finite, we henceforth assume that $\ker \mu = \langle 1 \rangle$, and identify G with $G\mu$. The statement of 3.4 can then be written more simply as $\operatorname{cl}_G(g^G) = \operatorname{cl}_{G^+}(g^{G^+}) \cap G$. What is the point of this result? Let g and g' be elements of G which are conjugate in every finite image of G. By 2.1 this means that for every $P \in I$ there exists $w_P \in G$ such that $g'\pi_P = (w_P^{-1}gw_P)\pi_P$, and at first glance it is conceivable that the "length" of the elements w_P (in whatever sense) might be unbounded. Now if we have $\operatorname{cl}_{G^+}(g^{G^+}) = g^{G^+}$, then 3.4 implies that $g' \in \operatorname{cl}_G(g^G) = g^{G^+} \cap G$, so $g' = w^{-1}gw$, where $w \in G^+$. This, of course, means that the above w_P can be chosen to have bounded length. It turns out that conjugacy separability is commonplace for elements of G^+. Before we proceed with this, we state the following, which does not require $\ker \mu = \langle 1 \rangle$.

Corollary 3.5. *If g is conjugacy distinguished in G, then*

$$(g\mu)^{G^+} \cap G\mu = g^G\mu.$$

Proof. For if $(g\mu)^{G^+} \cap G\mu = S\mu$, where $S \neq g^G$, then

$$g^G\mu = \operatorname{cl}_G(g^G)\mu = \operatorname{cl}_{G^+}((g\mu)^{G^+}) \cap G\mu \supseteq (g\mu)^{G^+} \cap G\mu = S\mu,$$

and so $\operatorname{cl}_G(g^G) \supseteq S \supseteq g^G$. □

It is necessary to study conjugacy separability of the base group of G^+ first. The next section is devoted to the study of $A^+ = \pi_1(\mathscr{G}|_Y, Y)$, where Y is a maximal subtree of X.

4. Conjugacy separability of A^+

We begin with the following general fact.

Proposition 4.1. *Let $\mathscr{G}$ be a graph of groups over X, and assume that*

$$(1) \qquad\qquad \bigcap_{P \in I} H_e P_{i(e)} = H_e \quad \text{for all } e \in EX,$$

with $G = \pi_1(\mathscr{G}, X)$. Let Y be a connected subgraph of X, and put $G(Y) = \langle A_x : x \in VY \rangle \leq G$. Then $G(Y) = \bigcap_{P \in I} G(Y) \ker \pi_P$. If Y is finite then $G(Y) = \bigcap_{M \triangleleft_f G} G(Y)M$.

Proof. Fix a vertex $x_0 \in VY$, and for each edge $e \in EX$ let $T_e \ni 1$ be a left transversal of H_e in $A_{i(e)}$. Then every element of G is uniquely represented by a normal word $t_1 \cdots t_n a$, where $a \in A_{x_0}$, $t_i \in T_{e_i}$ for $1 \leq i \leq n$, $(e_1, \ldots, e_n)$ is a closed path at x_0, and if $e_i = \bar{e}_{i-1}$ then $t_i \neq 1$ [8, Corollary 1]. Moreover, the proof shows that the elements of $G(Y)$ are represented by paths that entirely belong to Y. Now condition (1) means that, given any element $g = t_1 \cdots t_n a$ in normal form in G, we can find $P \in I$ such that $(t_1 \pi_P) \cdots (t_n \pi_P)(a \pi_P)$ is the normal form of $g \pi_P$ in G_P (relative to suitable transversals of the $H_e \pi_P$ in the $A_{i(e)} \pi_P$). In particular, if $g \in G \backslash G(Y)$ then g is represented by a path that goes outside Y, and since $g \pi_P$ is represented by the same path, we have $g \pi_P \notin G_P(Y) = \langle A_x \pi_P : x \in VY \rangle = G(Y) \pi_P$. The first part follows.

Now suppose that Y is finite. Then for any $P \in I$ we have the finitely generated subgroup $G(Y) \pi_P$ of the free-by-finite group G_P. Such subgroups are profinitely closed (e.g., [2, p. 229]). This evidently means that $G(Y) \ker \pi_P = \bigcap_{M \in P\uparrow} G(Y)M$, and the second part follows (cf. the proof of 2.1). $\square$

To simplify the notation let Y be a tree, $\mathscr{G}$ a graph of groups over Y, and $A = \pi_1(\mathscr{G}, Y)$. Also write $C = \bigcup_{x \in VY} A_x$ and $D = \bigcup_{e \in EY} H_e$, the subsets of *vertex elements* and *edge elements* respectively in A.

Proposition 4.2. *Let g be an element of A. Then the following assertions are true:*

(i) *If $g \in D$, then there exists a subtree Y_g of Y such that $g \in H_e$ if and only if e is an edge of Y_g.*

(ii) *If $g \notin D$, then there exists a FINITE subtree Y_g of Y such that $g \in A(Y_g)$, and if Z is any subtree of Y with $g \in A(Z)$, then $Y_g \subseteq Z$.*

Proof. (i) Let Y_g consist of all vertices x of Y such that $g \in A_x$, and all geodesics between them. If $g \in A_x \cap A_y$, and p is the Y-geodesic from x to y, then g belongs to every vertex group of p (since adjacent vertex groups in A generate their amalgamated free product). In other words, g belongs to every vertex group of Y_g. Moreover, $H_e = A_{i(e)} \cap A_{t(e)}$ for any edge e, so $g \in H_e$ if and only if $i(e)$ and $t(e)$ belong to Y_g.

(ii) Among all subtrees Z with $g \in A(Z)$ pick one with the fewest number of vertices, and call it Y_g. Fix a vertex x_0 in Y_g. Then the normal form $t_1 \cdots t_n a$ of g can be represented by a closed path p at x_0, with p entirely contained in Y_g (since $g \in A(Y_g)$). Now let Z be a subtree such that $g \in A(Z)$. Suppose first that $Y_g = \{x_0\}$. If $x_0 \notin VZ$ then $g \in A_{x_0} \cap A(Z)$ must be an edge element (consider the geodesic from x_0 to $A(Z)$), a contradiction. So in this case $Y_g \subseteq Z$.

There remains the case where Y_g has more than one vertex. If Z contains every end vertex of Y_g then $Y_g \subseteq Z$, so suppose Z does not contain some end vertex x of Y_g, and let e be the edge of Y_g that ends at x. Now the path p must visit x at least once (since otherwise $g \in A(Y_g \backslash \{e, x\})$), and if $t_i \in A_x$ then $t_i \neq 1$ (since p looks like $\ldots, e, x, \bar{e}, \ldots$ at x, and we have a normal form). If Z and Y_g were disjoint then $g \in A(Z) \cap A(Y_g)$ would be an edge element, which is not the case. Thus $Z' = Z \cup Y_g \backslash \{e, x\}$ is a subtree of Y such that $g \in A(Z')$. If x_0 is on Z' then we have the immediate contradiction that g can be represented in normal form by a path q in Z', and $q \neq p$ since x is on p but not on q. So we must have $x_0 = x$. But since Z and Y_g are not disjoint we can always choose x_0 to be a common vertex. This final contradiction shows that $Z \supseteq Y_g$, as required. $\square$

To define the notion of a reduced form for elements of A we need a definition. Let Δ' be a nonempty subtree of a finite tree Δ. By a *reduction process* R from Δ to Δ' we mean a sequence $\Delta_1, \ldots, \Delta_m$ of subtrees of Δ such that $\Delta_1 = \Delta$, $\Delta_m = \Delta'$, and Δ_{i+1} is obtained from Δ_i by deleting an end vertex x_i (for $i = 1, \ldots, m-1$). We let y_i be the vertex of Δ_i adjacent to x_i, so when $\Delta \subseteq Y$ we have

$$A(\Delta_i) = \langle A(\Delta_{i+1}) * A_{x_i} : H_{y_i x_i} = H_{x_i y_i} \text{ via } \theta_{y_i x_i} \rangle, \qquad i = 1, \ldots, m-1.$$

Let $\Delta \supseteq \Delta'$ be finite subtrees of Y, and $R = \{\Delta_1, \ldots, \Delta_m\}$ a reduction process from Δ to Δ'. Say g is reduced (resp. cyclically reduced) relative to R if either $g \in A(\Delta')$ or $g \in A(\Delta_i) \backslash A(\Delta_{i+1})$ for some $i \leq m-1$, and then g is reduced (resp. cyclically reduced) in the amalgamated free product $A(\Delta_i)$ of $A(\Delta_{i+1})$ and A_{x_i}. Every element of $A(\Delta)$ can be written in reduced form, and is conjugate, in $A(\Delta)$, to a cyclically reduced element of $A(\Delta)$. Note that an elementm may be cyclically reduced relative to one reduction process, but not another.

Lemma 4.3. *Assume that* (1) *holds. Let* g *be a nonvertex element of* A *such that* g *is cyclically reduced relative to some reduction process from* Y_g *to a single vertex. Then* $\mathrm{cl}_A(g^A) = \bigcup_\alpha g_\alpha^A$, *where each* $g_\alpha \in A(Y_g)$.

Proof. There exists $P_0 \in I$ such that $Y_{g\pi_P} = Y_g$ for all $P \subseteq P_0$. To begin with, we can choose P_1 such that $g\pi_P$ is cyclically reduced in $A(Y_g)\pi_P$ relative to R, for all $P \subseteq P_1$. This is because (1) can be used to ensure that the finitely many vertex elements in the cyclically reduced form of g are excluded from the finitely many edge subgroups encountered by R. In particular, if $P \subseteq P_1$ then $g\pi_P$ is not an edge element of $A\pi_P$. For such P we have $Y_{g\pi_P} \subseteq Y_g$ (since $g\pi_P \in A(Y_g)\pi_P$), and if $Q \subseteq P$ then $Y_{g\pi_Q} \supseteq Y_{g\pi_P}$. Since Y_g is finite, there exists a subtree $\Delta \subseteq Y_g$ and $P_0 \subseteq P_1$ such that for all $P \subseteq P_0$ we have $Y_{g\pi_P} = \Delta$. Then $g \in \bigcap_{P \subseteq P_0} A(\Delta) \ker \pi_P = A(\Delta)$ by 4.1, and then $\Delta = Y_g$ by definition of Y_g.

Now suppose $P \subseteq P_0$. Then $g\pi_P$ is a cyclically reduced nonvertex element of $A(Y_g)\pi_P$, so the conjugacy theorem for amalgamated free products [9] implies that the conjugacy class of $g\pi_P$ in $A(Y_g)\pi_P$ contains no vertex elements. If $\Delta \supseteq Y_g$ is a finite subtree then the conjugacy class of $g\pi_P$ in $A(\Delta)\pi_P$ contains no vertex elements (consider a reduction process from Δ to Y_g). Now

let $g' \in \mathrm{cl}_A(g^A)$, and let $\Delta \supseteq Y_g$ be a finite subtree with $g' \in A(\Delta)$. Let $R = \{\Delta_1, \ldots, \Delta_m\}$ be a reduction process from Δ to Y_g. Replacing g' by a conjugate in $A(\Delta)$ we may assume that g' is cyclically reduced relative to R, and we may choose P_0 such that $g'\pi_P$ is cyclically reduced for all $P \subseteq P_0$. If $P \subseteq P_0$ then $g\pi_P$ has minimal length in its $A(\Delta)\pi_P$-conjugacy class, and belongs to $A(\Delta_2)\pi_P \backslash H_{y_1 x_1}\pi_P$. The conjugacy theorem for amalgamated free products [9] implies that $g'\pi_P \in A(\Delta_2)\pi_P$. Continuing in this way we finally obtain $g'\pi_P \in A(Y_g)\pi_P$, whence $g' \in \bigcap_{P \subseteq P_0} A(Y_g)\ker \pi_P = A(Y_g)$, as required. $\square$

We can now prove

Theorem 4.4. *Let g be a nonvertex element of A such that g is cyclically reduced relative to some reduction process R from Y_g to a vertex. Assume that (1) holds and that $H_{x_1 y_1}$ is compact in the I-topology (where $x_1 y_1$ is the first edge deleted in the reduction process). Then g is conjugacy distinguished in A.*

Proof. By 4.3 we have $\mathrm{cl}_A(g^A) = \bigcup g_\alpha^A$, where each $g_\alpha \in A(Y_g)$, and we may assume that each g_α is cyclically reduced relative to R. Fix α, and choose P_0 such that for all $P \subseteq P_0$, $g\pi_P$, and $g_\alpha\pi_P$ are cyclically reduced in $A(Y_g)\pi_P$ relative to R. The conjugacy theorem for amalgamated free products can now be applied to deduce that $g_\alpha\pi_P$ is conjugate to some cyclic permutation of $g\pi_P$, via an element of $H_{x_1 y_1}\pi_P$. Replacing g by a cyclic permutation if necessary we have $g_\alpha\pi_P = (h_P^{-1}gh_P)\pi_P$ for all $P \subseteq P_0$, where $h_P \in H_{x_1 y_1}$. Consider the function $f: H_{x_1 y_1} \to A(Y_g)$ given by $f(h) = g_\alpha^{-1}h^{-1}gh$. With the appropriate profinitely topologies it is easy to see that f is continuous and $\ker \pi_P \cap A(Y_g)$ is closed in $A(Y_g)$. Thus for each $P \subseteq P_0$, $f^{-1}(\ker \pi_P)$ is a nonempty closed subset of $H_{x_1 y_1}$, and $P \subseteq Q$ implies that $f^{-1}(\ker \pi_P) \subseteq f^{-1}(\ker \pi_Q)$, so we have the finite intersection property. If

$$h \in \bigcap_{P \subseteq P_0} f^{-1}(\ker \pi_P) = f^{-1}\left(\bigcap_{P \subseteq P_0} \ker \pi_P\right) = f^{-1}(\langle 1 \rangle),$$

then $g_\alpha^{-1}h^{-1}gh = 1$, so $g_\alpha \in g^A$. The result follows. $\square$

The next lemma collects the information we need on conjugacy of edge and vertex elements.

Lemma 4.5. *Let $C = \bigcup_{x \in VY} A_x$ and $D = \bigcup_{e \in EY} H_e$.*

  (i) *Let $h \in D$, and let $a \in C$ be such that $h^a \in C$. Then $a \in A_x$ for some $x \in VY_h$.*

  (ii) *Let $a \in A_x$, $b \in A_y$, and $a^b \in C$. Then there exists a vertex z such that $a, b \in A_z$.*

  (iii) *Let $h \in D$, and w be a nonvertex element of A such that $h^w \in C$. Let $w = a_1 \cdots a_r$ be the reduced form of w (relative to some reduction process R of Y_w), where the $a_i \in C$. Then $h^{a_1 \cdots a_i} \in D$ for $1 \le i \le r-1$.*

  (iv) *Let $a \in A_x$ be such that $a^{A_x} \cap D = \varnothing$. Then $a^A \cap D = \varnothing$, and $a^A \cap C = a^{A_x}$.*

Proof. (i) Choose x as close to Y_h as possible subject to $a \in A_x$, and get a contradiction to $h^a \in C$ by assuming that $x \notin VY_h$ and considering the geodesic from x to Y_h.

(ii) Similar to (i): choose x_1 and y_1 as close to each other as possible (on the geodesic joining x and y) subject to $a \in A_{x_1}$ and $b \in A_{y_1}$.

(iii) First note that $Y_h \cap Y_w \neq \varnothing$, for otherwise a consideration of the geodesic from Y_h to Y_w gives the contradiction $h^w \notin C$. Thus $a = h^w \in A(Y_w)$. The proof is by induction on r, where $w = a_1 \cdots a_r$. Consider the first step in the reduction process: if h^{a_1} is not an edge element then h^w is reduced as written (in the amalgamated free product of $A(\Delta_2)$ and A_{x_1}) and so cannot be a vertex element. Therefore $h_1 = h^{a_1} \in D$, and $a = h_1^{a_2 \cdots a_r}$. The result follows by induction on r (using part (i)).

(iv) Suppose $a^w = h \in D$ for some nonvertex element $w = a_1 \cdots a_r$, in reduced form. Then $h^{a_r^{-1} \cdots a_1^{-1}} = a \in C$, and by part (iii) we get $a^{a_1} = h^{a_r^{-1} \cdots a_2^{-1}} \in D$. So $a^A \cap D \neq \varnothing$ implies that $a^b \in D$ for some vertex element b. By part (ii), a and b must belong to the same vertex group, which has to be A_x since a belongs to no other vertex group. But now we have $a^{A_x} \cap D \neq \varnothing$, a contradiction. It follows that $a^A \cap D = \varnothing$.

For the second part let $b = a^w \in A_y$, and choose y as close to x as possible subject to $b \in A_y$. Claim that $y = x$. For if not, then $Y_w \neq \{x\}$, and we have two cases to consider:

Case 1. $x \notin V Y_w$. Then there exists a finite subtree $\Delta \supseteq Y_w$ and an edge yx with $y \in V\Delta$. Now $a \in A_x \backslash H_{xy}$, $w \in A(\Delta)$, and $b = w^{-1}aw \in A(\Delta)$ (as Δ has to contain y because $b = a^w \in A(\Delta \cup \{x\})$ and $y \neq x$). This forces $w \in H_{yx} = H_{xy}$, and $a^w \in A(\Delta) \cap A_x = H_{xy}$, contradicting $a^A \cap D = \varnothing$.

Case 2. $x \in V Y_w$. Let $R = \{\Delta_1, \ldots, \Delta_m\}$ be a reduction process from Y_w to a point, such that the first vertex deleted is $x_1 \neq x$ (this can be done since $Yw \neq \{x\}$). Write $w = a_1 \cdots a_r$ in reduced form relative to R. Then $b = a_r^{-1} \cdots a_1^{-1} a a_1 \cdots a_r \in A_y \subseteq A(\Delta_2) \cup A(x_1)$, so we must have $a_1 \in A_x$ and $a^{a_1} \in H_{y_1 x_1}$ (since $r \geq 2$). This is a contradiction.

We have now shown that $b = a^w \in A_x$. An argument similar to that in Case 2 now shows that $w \in A_x$, so $a^A \cap C \subseteq a^{A_x}$, are required. $\square$

We can now prove

Theorem 4.6. *Let* $g \in A_x$ *be such that* $g^{A_x} \cap D = \varnothing$. *Assume that* X *is locally finite, that every edge subgroup of* A *is compact in the I-topology, and that* $\bigcap_{P \in I} g^{A_x} P_x = g^{A_x}$. *Then* g *is conjugacy distinguished in* A.

Proof. If $\mathrm{cl}_A(g^A)$ contains a nonvertex cyclically reduced element g_0, then by 4.4 we have $\mathrm{cl}_A(g^A) = \mathrm{cl}_A(g_0^A) = g_0^A$, so $g^A = g_0^A = \mathrm{cl}_A(g^A)$ and we are done. (Note that condition (1) is a consequence of the compactness assumption, so 4.4 is applicable.) We may therefore suppose that $\mathrm{cl}_A(g^A) = \bigcup a_i^A$, where each a_i is a vertex element of A. Let $S = \bigcup_{i(e)=x} H_e$, and note that S is compact since X is locally finite. Put $T_P = g^{A_x} \pi_P \cap S \pi_p$. We claim that there exists P_0 such that $T_P = \varnothing$ for all $P \subseteq P_0$. Suppose not. Note that $Q \subseteq P$ implies that T_Q maps into T_P under the obvious map (from A_Q to A_P), so if $T_P \neq \varnothing$ then we have an inverse system of nonempty finite sets. Let $(h_P \pi_P) \in \varprojlim T_P$, where each $h_P \in S$. Since S is compact there exists $h \in S$ such that $h \pi_P = h_P \pi_P$ for all P. This clearly means that $h \in \bigcap_{P \in I} g^{A_x} P_x = g^{A_x}$, contradicting $g^{A_x} \cap D = \varnothing$. Hence there exists P_0 such that $g^{A_x} \pi_P \cap D \pi_P = \varnothing$ for all $P \subset P_0$. By 4.5(iv)

we have $g^A \pi_P \cap D\pi_P = \varnothing$, and $a_i \pi_P \in g^A \pi_P \cap C\pi_P = g^{A_x} \pi_P$. Thus first of all $a_i \in \bigcap_{P \subseteq P_0} A_x \ker \pi_P = A_x$, and then $a_i \in \bigcap_{P \subseteq P_0} g^{A_x} P_x = g^{A_x}$. Thus $\text{cl}_A(g^A) = g^A$, as required. $\square$

Remark 1. For g as in 4.6, the condition $\bigcap_P g^{A_x} P_x = g^{A_x}$ is implied by $\text{cl}_A(g^A) = g^A$. For if $b \in \bigcap g^{A_x} P_x$, then $b \in \text{cl}_A(g^A) = g^A$, and so $b \in g^A \cap C = g^{A_x}$ by 4.5(iv).

Remark 2. The usual proof of the conjugacy separability of the profinite group $\widehat{A_x}$ in fact shows that for $g \in \widehat{A_x}$ we have $\bigcap_{P \in I} g^{\widehat{A_x}} \overline{P}_x = g^{\widehat{A_x}}$.

The next two results give partial information about conjugacy separability of edge elements.

Lemma 4.7. *Assume that* (1) *holds. Let* h *be an edge element of* A, *and assume that there exists* $P_0 \in I$ *such that* $Y_{h\pi_{P_0}} = Y_h$. *Then* $\text{cl}_A(h^A) \cap D = \text{cl}_{A(Y_h)}(h^{A(Y_h)}) \cap D$.

Proof. The condition implies that $h\pi_P \notin H_e\pi_P$ for all $P \subseteq P_0$, where e is an edge leading out of Y_h. Let $z \in \text{cl}_A(h^A) \cap D$, so $z\pi_P = h^w \pi_P$ for some $w \in A$. We claim that w can be chosen in $A(Y_h)$. For if not, let Δ be the smallest tree containing $h\pi_P$ and $w\pi_P$, and consider a reduction process $\{\Delta_1, \ldots, \Delta_m\}$ from Δ to $\Delta \cap Y_h$. Now the first vertex $x_1 \notin VY_h$, and $w\pi_P \notin A(\Delta_2)\pi_P$ by the minimal choice of Δ, and yet $(h\pi_P)^{w\pi_P}$ belongs to a vertex group. By the conjugacy theorem for amalgamated free product, we must have $h\pi_P \in H_{y_1 x_1}\pi_P$, contradicting the choice of P. We have therefore shown that

$$h_0 \in \bigcap_{P \subseteq P_0} h^{A(Y_h)} \ker \pi_P \subseteq \bigcap_{P \subseteq P_0} A(Y_h) \ker \pi_P = A(Y_h),$$

and so

$$h_0 \in \bigcap_{P \subseteq P_0} h^{A(Y_h)} \ker \pi_P \cap A(Y_h)$$

$$= \bigcap_{P \subseteq P_0} h^{A(Y_h)} (\ker \pi_P \cap A(Y_h)) = \text{cl}_{A(Y_h)}(h^{A(Y_h)}). \quad \square$$

Let h be an edge element of A. For any sequence $\sigma = (x_1, \ldots, x_r)$ of vertices of A put $D_\sigma(h) = D \cap \{h^{a_1 \cdots a_r} : a_i \in A_{x_i}\}$. We have

Theorem 4.8. *Assume that* X *is locally finite and* D *is compact. Let* h *be an edge element of* A^+. *Then* h *is conjugacy distinguished in* A^+ *if and only if* $h^{A^+} \cap D = D_\sigma(h)$ *for some finite sequence* σ *of vertices.*

Proof. We know that $\text{cl}_{A^+}(h^{A^+}) = \bigcup h_\alpha^{A^+}$ with each $h_\alpha \in D$. Let $K = h^{A^+} \cap D = \bigcup_\sigma D_\sigma(h)$, the union being over all finite sequences σ. If $z \in \text{cl}_{A^+}(h^{A^+}) \cap D$ and $P \in I$ then there exists a sequence $\sigma(P)$ such that $z\pi_P \in D_{\sigma(P)}(h)\pi_P \subseteq K\pi_P$, so $z \in K \ker \pi_P$. Thus

$$\text{cl}_{A^+}(h^{A^+}) \cap D \subseteq \bigcap_P K(\ker \pi_P) = \text{cl}_{A^+}(K).$$

Now D is a compact subset of A^+, and so is closed. Therefore $\text{cl}_{A^+}(K) = \text{cl}_D(K)$. Conversely if $z \in \text{cl}_D(K)$ and $P \in I$ then $z\pi_P \in K\pi_P$, so $z\pi_P \in$

$D_\sigma(h)\pi_P$ for some sequence σ (depending on P). In other words, $z \in \text{cl}_{A^+}(h^{A^+})$, and we have shown that $\text{cl}_{A^+}(h^{A^+}) \cap D = \text{cl}_D(h^{A^+} \cap D)$. We also note that for a fixed sequence σ, the set $D_\sigma(h)$ is a complete subspace of the compact set D (since each vertex group $\widehat{A}_x$ is compact), and is therefore compact. So if $h^{A^+} \cap D = D_\sigma(h)$ then $\text{cl}_{A^+}(h^{A^+}) \cap D = \text{cl}_D(K) = K = h^{A^+} \cap D$, which implies that every $h_\alpha \in h^{A^+}$, and so h is conjugacy distinguished in A^+. Conversely, suppose $\text{cl}_{A^+}(h^{A^+}) = h^{A^+}$. Then $K = h^{A^+} \cap D = \text{cl}_D(K)$, so the compact (since closed) set K is the (countable) union of the closed sets $D_\sigma(h)$. Enumerate the finite sequences $\{\sigma_1, \sigma_2, \dots\}$, and put $T_i = \bigcup_{1 \le j \le i} D_{\sigma_j}(h)$. Then $K = \bigcup_{i=1}^\infty T_i$, and each T_i is closed. By the Baire Category Theorem (or rather its proof, using compactness), some T_{i_0} has nonempty interior, and so contains a set of the form $hP_x \cap D$, where $h \in D$ and $P \in I$. Since $P_x \lhd_f A_x$ and K is compact, it is easy to see that K is the union of finitely many T_i, whence $K = T_j$ for some j. Let σ be the concatenation of the sequences $\sigma_1, \dots, \sigma_j$. Then it is clear that $D_{\sigma_i}(h) \subseteq D_\sigma(h)$ for $1 \le i \le j$, and so $h^{A^+} \cap D = K = T_j \subseteq D_\sigma(h)$, as required. $\square$

The following are immediate consequences of 4.7 and 4.8.

Corollary 4.9. *Let X be locally finite, and let h be an edge element of A^+ such that Y_h is finite. Then h is conjugacy distinguished in A^+ if and only if for some finite sequence σ of vertices of Y_h we have $h^{A^+} \cap D = D_\sigma(h)$.* $\square$

Corollary 4.10. *Let A^+ be the amalgamated free product of countably many groups. Then an element h of the associated subgroup H is conjugacy distinguished in A^+ if and only if $h^{A^+} \cap H = D_\sigma(h)$ for some finite sequence σ of vertices.* $\square$

5. CONJUGACY SEPARABILITY OF G^+

If X is not a tree then G^+ is a nontrivial HNN-extension over the base group $A^+ = \langle \widehat{A}_x : x \in VX \rangle$. By the length $|g|$ of an element $g \in G^+$ we mean the number of t-symbols appearing in an HNN-reduced form for g. Thus $g = a_1 t_{e_1}^{\varepsilon_1} \cdots t_{e_n}^{\varepsilon_n} a_{n+1}$, where all $a_i \in A^+$, a_{n+1} is arbitrary, each $\varepsilon_i = \pm 1$, each $e_i \in EX \backslash EY$ (where Y is a fixed maximal subtree of X), and if $e_{i-1} = e_i$ and $\varepsilon_{i-1} + \varepsilon_i = 0$ then $a_i \notin H_{\varepsilon_i e_i}$ (we write H_{-e} instead of $H_{\bar{e}}$ for ease of notation). The reduced element g is cyclically reduced if $a_{n+1} = 1$ and $t_{e_n}^{\varepsilon_n} a_1 t_{e_1}^{\varepsilon_1} \cdots a_n$ is also reduced.

Theorem 5.1. *Let $g \in G^+$ be cyclically reduced and of length at least one. Then g is conjugacy distinguished in G^+.*

Proof. Let z be a cyclically reduced element in $\text{cl}_{G^+}(g^{G^+})$. It is easy to see that the edge subgroups $\overline{H}_{\varepsilon e}$ are closed in A^+, so we can find $P_0 \in I$ such that $|g| = |g\pi_P|$, $|z| = |z\pi_P|$, and both $g\pi_P$ and $z\pi_P$ are cyclically reduced in G_P for all $P \subseteq P_0$. The conjugacy theorem for HNN-extensions (e.g. [3]) now implies that $|g\pi_P| = |z\pi_P|$, so $|g| = |z|$. Modulo the same P we know that $g\pi_P$ and some cyclic permutation of $z\pi_p$ have the same sequence of t-symbols with the same exponents. Replacing z by a suitable cyclic permutation we may assume that $g = a_1 t_{e_1}^{\varepsilon_1} \cdots a_n t_{e_n}^{\varepsilon_n}$ and $z = b_1 t_{e_1}^{\varepsilon_1} \cdots b_n t_{e_n}^{\varepsilon_n}$. For $P \subseteq P_0$ the conjugacy theorem in G_P implies the existence of elements $h_{0,P}, \dots, H_{n,P}$ such that, modulo P

$$h_{0,P}b_1 \equiv a_1 h_{1,P}, \qquad h_{1,P} \in \overline{H}_{\varepsilon_1 e_1},$$

$$(h_{1,P}\overline{\theta}_{e_1}^{\varepsilon_1})b_2 \equiv a_2 h_{2,P}, \qquad h_{2,P} \in \overline{H}_{\varepsilon_2 e_2},$$

$$\vdots$$

$$(h_{n-1,P}\overline{\theta}_{e_{n-1}}^{\varepsilon_{n-1}})b_n \equiv a_n h_{n,P}, \qquad h_{n,P} \in \overline{H}_{\varepsilon_n e_n},$$

$$(h_{n,P}\overline{\theta}_{e_n}^{\varepsilon_n}) \equiv h_{0,P}, \qquad \text{so } h_{0,P} \in \overline{H}_{-\varepsilon_n e_n} = \overline{H}_{\varepsilon_0 e_0}.$$

Since $\bigcup_{i=0}^{n} \overline{H}_{\varepsilon_i e_i}$ is compact, there exist elements $h_i \in \overline{H}_{\varepsilon_i e_i}$ such that $h_i \pi_P = h_{i,P}\pi_P$ for all P. Then we have the equations $h_0 b_1 = a_1 h_1$, $(h_1 \overline{\theta}_{e_1}^{\varepsilon_1})b_2 = a_2 h_2, \ldots, h_n \overline{\theta}_{e_n}^{\varepsilon_n} = h_0$, and so $z = h_0^{-1}gh_0$. The result follows. $\square$

To deal with conjugacy separability of elements of A^+ in G^+, we begin with the following trivial observation, which can be proved by using a length argument (as in the proof of 5.1) and the conjugacy theorem for HNN-extensions.

Lemma 5.2. *If* $a \in A^+$ *then* $\mathrm{cl}_{G^+}(a^{G^+}) = \bigcup a_\alpha^{G^+}$, *where each* $a_\alpha \in A^+$. *In particular, a is conjugacy distinguished in G^+ if and only if* $\mathrm{cl}_{G^+}(a^{G^+}) \cap A^+ = a^{G^+} \cap A^+$. $\square$

The conjugacy theorem for elements of the base of an HNN-extension states that if $a, b \in A^+$ are conjugate in G^+, then there exist elements $h_1, \ldots, h_m \in \bigcup_{e \in EX \backslash EY} \overline{H}_e$, such that $a \sim h_1$ in A^+ (i.e., conjugate in A^+), $b \sim h_m$ in A^+, and for $1 \le i \le m-1$, either $h_{i+1} \sim h_i$ in A^+ or $h_{i+1} = h_i \theta_e$ for some $e \in EX \backslash EY$. Now we have

Proposition 5.3. *Let* $a \in A^+$ *be a nonvertex cyclically reduced element (relative to some reduction process of Y_a). Then* $\mathrm{cl}_{G^+}(a^{G^+}) \cap A^+ = a^{A^+}$, *and so a is conjugacy distinguished in G^+.*

Proof. Choose P_0 such that for all $P \subseteq P_0$ we have $Y_{a\pi_P} = Y_a$ and $a\pi_P$ is cyclically reduced (cf. the proof of 4.4). Then $(a\pi_P)^{A^+\pi_P} \cap D\pi_P = \varnothing$ by 4.5(iii) and using the fact that no nonvertex conjugate of an edge element can be cyclically reduced. Now let $b \in \mathrm{cl}_{G^+}(a^{G^+}) \cap A^+$. Then $b\pi_P$ is conjugate to $a\pi_P$ in G_P. Since no $A^+\pi_P$-conjugate of $a\pi_P$ can belong to an edge subgroup in G_P, we must have $b\pi_P$ conjugate to $a\pi_P$ in $A^+\pi_P$, and so $b \in \mathrm{cl}_{A^+}(a^{A^+}) = a^{A^+}$, the latter by 4.4. The final part follows from 5.2. $\square$

For vertex elements of A^+ we have the following (recall that $C = \bigcup_{x \in VX} \widehat{A}_x$):

Proposition 5.4. *Assume that X is locally finite. Let a be a vertex element of A^+, and let $b \in \mathrm{cl}_{G^+}(a^{G^+}) \cap C$. Then there exists a sequence $c_0, c_1, \ldots$ of elements of C such that* (i) $c_0 = a$; (ii) *for each* $i \ge 1$, *either* $c_i \sim c_{i-1}$ *in* C, *or* $c_i = c_{i-1}\theta_{e_i}$ *for some* $e_i \in EX \backslash EY$; *and* (iii) $c_i \to b$ *as* $i \to \infty$.

Proof. For $P \in I$ let $S_P(a, b)$ denote the set of all sequences $\sigma = \{a_{i,P}\pi_P : i \ge 0\}$, where each $a_{i,P} \in C$, $a_{0,P}\pi_P = a\pi_P$, $a_{i,P}\pi_P = b\pi_P$ for all sufficiently large i, and each $a_{i,P}\pi_P$ is either $C\pi_P$-conjugate to, or the $\theta_{e,P}$-image of, $a_{i-1,P}\pi_P$ (in other words, each $\sigma \in S_P(a, b)$ represents a possible conjugacy $a\pi_P \sim b\pi_P$ in G_P^+). For fixed i and P write $T_i(P)$ for the set of ith components of all $\sigma \in S_P(a, b)$. Suppose $a \in \widehat{A}_x$, and let Δ_i denote the (finite)

subtree of radius i with centre x. A simple inductive argument shows that $T_i(P) \subseteq \bigcup_{y \in V\Delta_i} \widehat{A}_y \pi_P$, so each $T_i(P)$ is finite. If $Q \subseteq P$ then every sequence in $S_Q(a, b)$ maps to a sequence in $S_P(a, b)$ under the induced map $G_Q^+ \to G_P^+$, and so $T_i(Q)$ maps into $T_i(P)$.

We construct the elements c_i inductively, beginning with $c_0 = a$. Assume that $c_1, \ldots, c_{i-1}$ have been chosen subject to (ii) and $c_j \pi_p \in T_j(P)$ for all P, and $1 \leq j \leq i - 1$. Let $K_P = \{c' \pi_P \in T_i(P) :$ there exists $\sigma \in S_P(a, b)$ such that $\sigma_{i-1} = c_{i-1} \pi_P$ and $\sigma_i = c' \pi_P\}$. Then each $K_P \neq \varnothing$, and if $Q \subseteq P$ then K_Q maps into K_P under the induced map $G_Q \to G_P$. Let $(d_P \pi_P)$ belong to the inverse limit of the sets K_P. Then each $d_P \in \bigcup_{y \in V\Delta_i} \widehat{A}_y$, and the latter set is compact, so there exists $c_i \in \bigcup_{y \in V\Delta_i} \widehat{A}_y$ such that $c_i \pi_P = d_P \pi_P$ for all P. Therefore, for every P there exists $\sigma \in S_P(a, b)$ such that $\sigma_{i-1} = c_{i-1} \pi_P$ and $\sigma_i = c_i \pi_P$. Also $c_i \pi_P \in T_i(P)$ for all P. To verify (ii), note that for each P the element $c_i \pi_P$ is either conjugate to $c_{i-1} \pi_P$ by some element of $\bigcup_{y \in V\Delta_{i-1}} \widehat{A}_y \pi_P$, or is the image of $c_{i-1} \pi_P$ under some $\theta_{e,P}$, where $e \in E\Delta_i$. The number of possibilities is therefore finite and independent of P, and so at least one must hold for all P. If $c_i \pi_p = c_{i-1} \pi_P \theta_{e,P}$ for all P, then $c_{i-1} \in \overline{H}_e$ and we get $c_i \pi_P = (c_{i-1} \theta_e) \pi_P$ for all P, whence $c_i = c_{i-1} \theta_e$. If $c_i \pi_P \sim c_{i-1} \pi_P$ in $\bigcup_{y \in V\Delta_i} \widehat{A}_y \pi_P$ for all P, then $c_i \sim c_{i-1}$ in $\bigcup_{y \in V\Delta_i} \widehat{A}_y$, using compactness. This proves (ii) for c_i, and so the inductive construction goes through. Finally given P it is clear that $c_i \pi_P = b \pi_P$ for all sufficiently large i, so $c_i \to b$ as $i \to \infty$. $\square$

Corollary 5.5. *Let $a \in A^+$ be a vertex element. Then a is conjugacy distinguished in G^+ if and only if $a^{G^+} \cap C$ is a closed subset of A^+.*

Proof. By 5.2 and the proof of 5.3 we see that $\mathrm{cl}_{G^+}(a^{G^+}) = \bigcup a_\alpha^{G^+}$, where each $a_\alpha \in C$. By 5.4, for each a_α there exists a sequence $c_i \in C$ such that $c_i \sim a$ in G^+ and $c_i \to a_\alpha$ as $i \to \infty$. Thus each $a_\alpha \in \mathrm{cl}_{A^+}(a^{G^+} \cap C)$, and so $\mathrm{cl}_{G^+}(a^{G^+}) \cap C \subseteq \mathrm{cl}_{A^+}(a^{G^+} \cap C)$. The reverse inclusion being trivial, we have $\mathrm{cl}_{G^+}(a^{G^+}) \cap C = \mathrm{cl}_{A^+}(a^{G^+} \cap C)$. The result follows. $\square$

Corollary 5.6. *Let a be a vertex element of A^+, and let $D' = \bigcup_{e \in EX \setminus EY} \overline{H}_e$. If $a^{A^+} \cap D' = \varnothing$, then a is conjugacy distinguished in G^+ if and only if a is conjugacy distinguished in A^+.*

Proof. Since no A^+-conjugate of a belongs to an associated subgroup, we have $a^{G^+} \cap A^+ = a^{A^+}$ by the conjugacy theorem for HNN-extensions, and so $a^{G^+} \cap C = a^{A^+} \cap C$. The result follows from 5.5 and the fact that $\mathrm{cl}_{A^+}(a^{A^+}) = \bigcup a_\alpha^{A^+}$, with all $a_\alpha \in C$ (this is a consequence of 4.4). $\square$

To deal with elements of $D' = \bigcup_{e \in EX \setminus EY} \overline{H}_e$ we need the following:

Lemma 5.7. *If D is compact, then so is $C = \bigcup_{x \in VX} \widehat{A}_x$.*

Proof. It is sufficient to prove that C is complete, since we can always regard C as a subspace of the profinite completion $\widehat{A}^+$. So let $\{a_P : P \in I\}$ be a subset of C such that for all $Q \subseteq P$ we have $a_Q \pi_P = a_P \pi_P$. If every $a_P \in D$, then the compactness of D implies the existence of $h \in D$ such that $a_P \pi_P = h \pi_P$ for all P, and we are done. Now suppose that $a_{P_0} \pi_{P_0} \notin D \pi_{P_0}$ for some P_0. Then $a_P \pi_P \notin D \pi_P$ for all $P \subseteq P_0$, for if $a_p \in \widehat{A}_y \neq \widehat{A}_x$ for some $P \subseteq P_0$,

then $a_{P_0} \pi_{P_0} = a_P \pi_{P_0} \in \widehat{A}_y \pi_{P_0} \cap \widehat{A}_x \pi_{P_0} \subseteq D \pi_{P_0}$, contrary to assumption. But now the compactness of $\widehat{A}_x$ gives the result. $\square$

Consider the set of all finite sequences $\sigma = (x_1, e_1^{\varepsilon_1}, \ldots, e_n^{\varepsilon_n}, x_{n+1})$, where $x_i \in VX$, $e_i \in EX \backslash EY$, and $\varepsilon_i = \pm 1$. For such a sequence let

$$B_\sigma(h) = C \cap \{h^g : g = a_1 t_{e_1}^{\varepsilon_1} \cdots t_{e_n}^{\varepsilon_n} a_{n+1}, \ a_i \in \widehat{A}_{x_i} \text{ for all } i\}.$$

Theorem 5.8. *Assume that X is locally finite and D is compact. Then $h \in D'$ is conjugacy distinguished in G^+ if and only if there exist finite sequences $\sigma_1 \ldots, \sigma_m$ such that $h^{G^+} \cap C = \bigcup_{i=1}^m B_{\sigma_i}(h)$.*

Proof. By 5.5 the element h is conjugacy distinguished in G^+ if and only if $K = h^{G^+} \cap C$ is closed in A^+. Since C is compact (by 5.7) and therefore closed, this is equivalent to K being closed in C, i.e., to K being compact. Now $K = \bigcup_\sigma B_\sigma(h)$, the union being over the countably many sequences σ. An application of Baire's Category Theorem (cf. the proof of 4.8) shows that K is compact if and only if it is the union of finitely many $B_{\sigma_i}(h)$. (One needs to show that each $B_\sigma(h)$ is closed. For this, use the compactness of the $\widehat{A}_x$ to show that $B_\sigma(h)$ is complete.) $\square$

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